CRACK GROWTH UNDER CYCLIC LOADS

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A large proportion of failures of machine parts is associated with the propagation of fatigue cracks. In such cases the "life" of a part counted from the instant of crack nucleation constitutes a considerable part of its total time-to-rupture. Fatigue fracture, which in the USA alone is being investigated in more than 200 laboratories, has been studied by many investigators both in the Soviet Union (N. N. Davidenkov, S. V. Serensen, I. A. Oding, R. M. Schneiderovich, R. D. Vagapov) and elsewhere (Orowan, Frost, McClintock, Paris). However, insufficient attention has been paid to quantitative studies of the growth of fatigue cracks.

In this article a phenomenological description of the process of crack propagation under the influence of cyclic loads is presented. The propagation of cracks in elastic-plastic solids under the influence of monotonically increasing loads is discussed first with particular reference to the determination of one of the size effects and the phenomenon of discontinuous crack growth (section 1). In section 2 cyclic loading conditions are discussed; the application of the Irwin-Orowan energy concept made it possible to derive a simple expression for the crack propagation rate which is in good agreement with experimental data. A phenomenological approach to the problem of nonpropagating cracks is described in section 3, while the problem of stability of crack propagation is analyzed in section 4. Finally, certain concrete problems are discussed in section 5.

The fracture of a specimen under the influence of cyclic loads can be described in general terms as follows [1–5]. At first no noticeable changes take place. Then, after a certain number of stress cycles, dislocations and submicroscopic cracks are formed in the material, which becomes slightly weaker. In the next stage microscopic cracks appear; the material continues to lose its cohesion and local plastic flow takes place. The final stage is characterized by the formation of a macroscopic crack leading to brittle fracture [6–8]. It may be taken as an established fact that the macrocrack grows during each stress cycle [9, 10]; fractographic examination of the fracture surfaces reveals the presence of characteristic furrow-like formations resembling annual growth rings in tree trunks.

In formulating a phenomenological description of fatigue fracture it is convenient to divide the process into two stages. In the first stage, the size of the dislocation and microcracks formed is comparable to the linear dimensions of regions of strength heterogeneities (i.e., grains); at this stage, (i.e., the crack nucleation stage) it is necessary to take into account the microstructure and heterogeneity of the material. The second stage is characterized by the growth of one (most dangerous) macrocrack whose size is large in comparison to the grain size; when the crack propagation rate at this stage is considered, the heterogeneity of the material may therefore be neglected and the material may be regarded as homogeneous and isotropic. Only the latter stage (i.e., the crack propagation stage) is considered below. However, the approach used can be applied in the analysis of transcrystalline and intercrystalline microcracks or dislocations; in this case the pertinent constants (e.g., yield point or dissipation energy) assume, naturally, different values appropriate to the material in the grain interior or in the grain-boundary regions.

Opinions about the relative parts played by the above two stages in fatigue fracture differ; the majority view appears to be that the crack propagation stage constitutes the larger part of the total time-to-rupture of a fatigue test piece [1, 3, 5, 11–13].

1. CRACK PROPAGATION IN ELASTIC-PLASTIC BODIES UNDER THE INFLUENCE OF MONOTONICALLY INCREASING LOADS

A crack in an ideal elastic body begins to grow only after the stress intensity coefficient N at the crack edge has reached the value of Irvin's constant K_c [14–16], the equality $N = K_c$ being satisfied during the quasi-static crack propagation. It is clear that no crack propagation can take place in an ideally elastic body under the influence of cyclic loads.

To explain the growth of fatigue cracks, it is necessary to resort to an elastic-plastic model of a solid. Let us therefore first consider the propagation of cracks in an elastic-plastic body under the influence of monotonically increasing loads; in these circumstances a plastic region with a characteristic linear dimension d exists near the crack tip. Let us confine our analysis to the most typical and general case in which the size of the plastic region is small in comparison with the characteristic geometric dimension of the body in question (e.g., the crack length). However, the general approach used below is applicable even when this condition is not satisfied (see, for instance, the elastic-plastic analogue of the Griffith problem in [17]). When this condition is satisfied, it is possible to introduce the concept of the stress intensity coefficient N; this coefficient determines the stress and strain distribution at distances from the crack tip which are large in relation to d but small in comparison with the characteristic dimension of the body; it is found from a solution of the purely elastic problem as a whole [18]. The following analysis is concerned with the fine structure of the crack tip (Fig. 1).

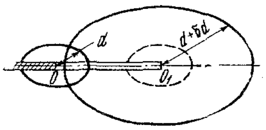

Fig. 1

Condition on the crack edge in an elastic-plastic body. Let σ_s denote the yield stress in tension. The size of the plastic region d can depend only on N, σ_s, Young's modulus E, and Poisson's ratio ν. It should be born in mind that N has the dimensions of force divided by length to the power of 3/2. Dimensional analysis [19] gives us

$$d = \alpha_1\left(\nu, \frac{\sigma_s}{E}\right)\frac{N^2}{\sigma_s^2}. \tag{1.1}$$

Here $\alpha_1(\nu, \sigma_s/E)$ is a certain dimensionless function.

When the external load monotonically increases, N also monotonically increases inthe vicinity of every point along the crack edge, especially in the vicinity of point O (Fig. 1). It is assumed that at the instant of loading the body under consideration is in a stress-free state. With increasing N the crack, generally speaking, will also increase. Let γ_* denote the dissipation of energy (per unit surface area) due to crack propagation. The increase in the crack length Δl can obviously depend only on N, σ_s, γ_*, ν, and E. Dimensional analysis gives

$$\Delta l = \frac{N^2}{\sigma_s^2}\Phi\left(\frac{N^2}{E\gamma_*}, \frac{\sigma_s}{E}, \nu\right). \tag{1.2}$$

Here Φ is a certain dimensionless function of its arguments.

Let us derive the equation of energy (Fig. 1). When the crack length is increased by an infinitely small value δl, the total dissipating energy $2\gamma_*\delta l$ evidently consists of two components. The first component δE_1 is numerically equal to the liberated elastic energy; it reflects the fact that N (and, consequently, d in accordance with (1.1)) remained constant when the crack length was increased by δl. The value δE_1 can depend only on N, E, σ_s, δl, and ν. Using dimensional analysis, we obtain

$$\delta E_1 = 2\alpha_2\left(\frac{\sigma_s}{E}, \nu\right)\frac{N^2}{E}\delta l. \tag{1.3}$$

The average intensity of plastic deformation in the plastic region depends only on σ_s/E and ν is independent of N, since there is no characteristic dimension of the body in the case under consideration. This means that formula (1.3) can also be obtained with the aid of (1.1) on the basis of considerations according to which δE_1 represents the irreversible work of plastic strains due to displacement of the plastic region (regarded as rigid) in the direction of the crack propagation (Fig. 1).

The second component δA_p represents the irreversible work of plastic strains, which is associated with the increase in the extent of the plastic region during loading and which is not related to the crack propagation; it reflects the fact that the crack length remained constant when N was increased by δN (Fig. 1). The value $\delta A_p = T\delta S\Gamma_p$ (where T is tangential stress intensity, δS is the increase in the volume of the plastic region and Γ_p is the average plastic strain intensity in δS) can depend only on σ_s, E, N, ν, and δN. Since T can depend only on σ_s, δS only on δd, and Γ_p only on σ_s, E, and ν, using dimensional analysis and formula (1.1), we obtain

$$\delta A_p = 2\alpha_3\left(\frac{\sigma_s}{E}, \nu\right)\frac{N^3}{\sigma_s^3}\delta N. \tag{1.4}$$

Finally, we obtain the equation of energy in the form

$$\gamma_* = \alpha_2\left(\frac{\sigma_s}{E}, \nu\right)\frac{N^2}{E} + \alpha_3\left(\frac{\sigma_s}{E}, \nu\right)\frac{N^3}{\sigma_s^3}\frac{dN}{dl}. \tag{1.5}$$

Equation (1.5) should be especially valid in the case of ideal elastic bodies; in this case the second term on the right-hand side of (1.5) vanishes. In this limiting case one can, using Irvin's formula [14–16], find α_2:

$$\alpha_2 = \begin{cases} \pi & \text{(plane stress state)} \\ \pi(1-\nu^2) & \text{(plane strain)} \end{cases}. \tag{1.6}$$

Let us now apply the Irwin–Orowan physical concept, according to which γ_* represents a material constant [14–18]. Equation (1.5) can be written in a form resembling the expression for the flow of hardening elastic-plastic solids:

$$dl = \frac{\alpha_3 E N^3 dN}{\alpha_2\sigma_s^3(K_c^2 - N^2)} \qquad \left(K_c^2 = \frac{E\gamma_*}{\alpha_2}\right). \tag{1.7}$$

Here K_c is the Irwin constant.

Integrating (1.7), we find

$$l - l_0 = -\frac{\alpha_3 E K_c^2}{2\alpha_2\sigma_s^3}\left[\frac{N^2}{K_c^2} + \ln\left(1 - \frac{N^2}{K_c^2}\right)\right]. \tag{1.8}$$

$$(N = 0 \quad \text{at} \quad l = l_0)$$

The solid curve in Fig. 2 represents Eq. (1.8) plotted in dimensionless coordinates N_* and Δl_*:

$$\Delta l_* = \frac{2\alpha_2\sigma_s^3}{\alpha_3 E K_c^2}(l - l_0)], \qquad N_* = \frac{N}{K_c}, \tag{1.9}$$

Condition (1.8) plays the part of an additional boundary condition on the crack edge in an elastic-plastic body; after determining by plastic analysis of stresses the function $N = N(p, l)$, where p is the external load parameter, it is possible to find with the aid of Eq. (1.8) the dependence of l on p in any given concrete problem.

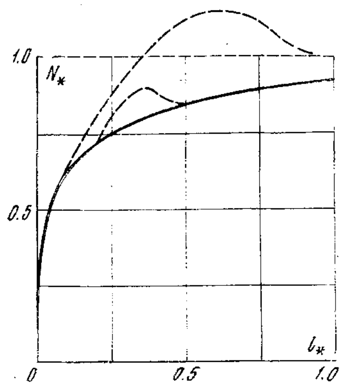

Fig. 2

As shown in Fig. 2, crack propagation in elastic-plastic bodies takes place also when $N < K_c$; the Irwin condition $N = K_c$ is satisfied asymptotically at $\Delta l_* \gg 1$, when the starting conditions no longer affect the issue; in practice, according to data in Fig. 2, any elastic-plastic body begins to behave like an ideal brittle body already at $\Delta l_* \gtrsim 2$.

The above qualitative singularities of crack propagation in elastic-plastic bodies are well known to experimental investigators [20–22].

The quantities α_1, α_2, and α_3 appearing in (1.1) and (1.9) remained indeterminate; they can be found from experimental data. In the case of through cracks in plates of elastic-plastic materials which satisfy the Tresk-St. Venant plasticity condition there is an approximate solution of the elastic-plastic problem [23,17]. Calculation gives in this case the following values of α_1 and α_3 [18]:

$$\alpha_1 = \frac{\pi^2}{4}, \qquad \alpha_3 = \frac{\pi^3 \sigma_s}{3E}. \tag{1.10}$$

The plastic region represents a segment of a length d on the extension of the crack.

Size effect. It has long been known that a given material can behave quite differently in different structures; it may behave as a ductile material in thin-walled or small-size structures and as a rather brittle material in thick-walled or large-size structures. This effect can be explained as a consequence of the above-outlined theory. According to formula (1.1), the maximum possible size d of the plastic region near a crack tip is

$$d_{\max} = \alpha_1 \frac{K_c^2}{\sigma_s^2}. \tag{1.11}$$

For the sake of simplicity let us consider a specimen with a crack (Fig. 3). If $d_{max} \ll L$, the material will behave during the process of fracture as a brittle material, i.e., the ultimate load will substantially depend on the initial crack length; if, however, $d_{max} \gtrsim L$, the material will behave as a completely ductile material and the ultimate load will be only slightly affected by the initial crack length; in the limiting case of a hard-ductile body, i.e., when $d_{max} \gg L$, the ultimate load will be independent of the initial crack length and will be entirely determined by the effective cross section of the specimen. Since the presence of initial cracks or defects in any given material is unavoidable, and since their size is random in character and difficult to control, it is obvious that in the case of structures carrying tensile loads the designer will give preference to more ductile, even if less strong, materials.

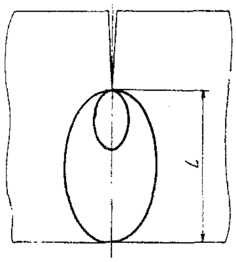

Fig. 3

It follows from the foregoing that in the rational selection of structural materials one of the basic characteristics considered (in addition to strength and pliability) should be the degree of reliability of the material measured in terms of dimensionless parameter

$$\chi = \frac{K_c^2}{\sigma_s^2 L}. \tag{1.12}$$

Here L denotes the characteristic linear dimension of the structure in question. When $\chi \gg 1$, the material behaves as ductile and the initial defects may be neglected in the design calculations. When $\chi \sim 1$ and especially when $\chi \ll 1$, it becomes necessary to take account of the internal material defects and to apply the theory of cracks. It may be stated that, generally speaking, the larger χ, the more reliable the material, other factors (especially the number and size of internal defects) being equal.

Discontinuous crack growth. The Irwin-Orowan concept of constant γ_* provides an accurate description of continuous (gradual) growth of cracks in elastic-plastic bodies (Fig. 2). However, there is a second-order effect, i.e., discontinuous crack propagation in certain elastic-plastic materials, which cannot be explained within the framework of this concept. This phenomenon corresponds to the presence of a hump on the curve* $N(\Delta l)$ shown in Fig. 2 by a dashed line and is analogous to the delay in plastic flow (a "tooth" on the $\sigma(\varepsilon)$ diagram). The $N(\Delta l)$ diagram is in its physical sense analogous to the $\sigma(\varepsilon)$ diagram and, in accordance with (1.2), can be found directly by experiment without bringing in any additional physical concepts. The presence of a hump on this diagram is evidently due to the presence (in heterogeneous material structure) of fairly strong components inhibiting the growth of cracks and dislocations; the height of the hump (as the height of the tooth [24]) is substantially dependent on the rate of

*Similar diagrams in articles by Ya. B. Fridman et al. were aptly called fracture diagrams.

loading, i.e., on dN/dt, where t denotes time.

2. CRACK GROWTH UNDER CYCLIC LOADS

Let us now consider the quasi-static crack propagation in elastic-plastic bodies in the case of cyclic loads which constitute certain periodic functions of time. If one formulates the problem of the fine structure of a crack tip for the case in which the concept of the stress intensity coefficient is meaningful, the propagation rate of the crack tip dl/dn under the influence of cyclic loads can depend only on the maximum and minimum values of the stress intensity coefficient during one stress cycle (N_{max} and N_{min}), on the number of stress cycles n, on the energy γ_* dissipated as a result of a unit surface area of the crack, and on material constants E, σ_s, and ν. Since n is very large, it can be regarded as a continuous argument. Dimensional analysis gives

$$\frac{dl}{dn}=\frac{N_{max}^2}{\sigma_s^2}\Psi\left(\frac{N_{max}^2}{E\gamma_*},\ \frac{N_{min}}{N_{max}},\ n,\ \frac{\sigma_s}{E},\ \nu\right). \tag{2.1}$$

Here Ψ is a certain dimensionless function of its arguments.

Crack propagation rate. The functional relation (2.1) can be determined on the basis of considerations substantially similar to those outlined in the previous chapter. It was assumed previously that the body under consideration is in a strain- and stress-free state at the instant of the application of the load; let us now analyze the process of crack propagation due to increasing N from N_{min} to N_{max}, it being assumed that events preceding the instant at which $N = N_{min}$ led to the appearance of residual (initial) stresses and strains in the body. It is easy to see that the entire reasoning process used in the derivation of formulas (1.1), (1.3)–(1.5), and (1.7) can be directly applied to the case under consideration; however, α_1, α_2, and α_3 now depend on the deformation history before the initial instant, at which $N = N_{min}$. Let σ_0 denote a parameter characterizing the magnitude of initial stresses responsible for the previous course of loading and crack propagation ($\sigma_0 \ll \sigma_s$).

In analogy to formulas (1.1) and (1.7) we obtain

$$\frac{dl}{dN}=\alpha_4\left(\frac{\sigma_0}{E},\ \frac{\sigma_s}{E},\ \nu\right)\frac{EN^3}{\sigma_s^3(K_c^2-N^2)},\quad d=\alpha_1\left(\frac{\sigma_0}{E},\ \frac{\sigma_s}{E},\ \nu\right)\frac{N^2}{\sigma_s^2}. \tag{2.2}$$

Here the material constants σ_s, E, K_c, and ν depend, generally speaking, on the previous deformation history. For the sake of simplicity, however, this dependence is neglected in the following analysis.

It can be shown that the dependence of functions α_1 and α_4 on the first argument σ_0/E may be neglected. This follows from the fact that $\sigma_0 \ll \sigma_s$ and σ_s/E is very small (~0.01) for all the structural materials; moreover, it is known from the previous section that at $\sigma_0 \to 0$ there are finite limits of continuous functions $\alpha_1(\sigma_0/E, \sigma_s/E, \nu)$ and $\alpha_4(\sigma_0/E, \sigma_s/E, \nu)$. Let β denote the following material constant*:

$$\beta=\alpha_4\left(0,\ \frac{\sigma_s}{E},\ \nu\right)\frac{EK_c^2}{2\sigma_s^3}. \tag{2.3}$$

It is quite natural to assume that no change in the crack length takes place when the load is reduced, i.e., when N_{max} decreases to N_{min}.

Integrating (2.2) between N_{min} and N_{max}, we obtain an expression for the increase in crack length Δl during one stress cycle:

$$\Delta l=-\beta\left(\frac{N_{max}^2-N_{min}^2}{K_c^2}+\ln\frac{K_c^2-N_{max}^2}{K_c^2-N_{min}^2}\right). \tag{2.4}$$

Transforming to continuous variables, we find the unknown rate of crack growth

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*Thus, the assumption that the influence of residual stresses on crack propagation may be neglected is in this case verified by the possibility of replacing $f(\varepsilon)$ by $f(0)$, since $f(0)$ is finite and ε small. This assumption is made in all the investigations known to the author of the growth of fatigue cracks, e.g., in papers by McLintock [25], Paris [26], and Hult [27].

$$\frac{dl}{dn} = -\beta\left(\frac{N_{max}^2 - N_{min}^2}{K_c^2} + \ln\frac{K_c^2 - N_{max}^2}{K_c^2 - N_{min}^2}\right). \tag{2.5}$$

It is convenient to determine the crack propagation rate as given by (2.5), (1.8), and (1.9) with the aid of the fracture diagram N_* versus Δl_* (Fig. 2).

Equation (2.5) plays the part of an additional boundary condition on the crack edge in an elastic-plastic body subjected to cyclic loads; the dependence of l on external load parameters p_{max} and p_{min} in any given concrete problem is found after substituting stress intensity coefficients N_{max} and N_{min} (determined by a purely elastic analysis and constituting functions of p and l) in the differential equation (2.5) and solving it in respect of $l(n)$.

Equation (2.5) can obviously be used to analyze also the case in which p_{max} and p_{min} vary with time (measured in terms of the number of cycles).

Let us replace the right-hand side of (2.5) by a segment of a Taylor series; this gives the following, sometimes more convenient equation:

$$\frac{dl}{dn} = \beta\left(\frac{N_{max}^4 - N_{min}^4}{2K_c^4} + \frac{N_{max}^6 - N_{min}^6}{3K_c^6} + \frac{N_{max}^8 - N_{min}^2}{4K_c^8} + \dots\right) \tag{2.6}$$

If $N_{min} < 0$, one should take $N_{min} = 0$ in (2.5) and (2.6), since in compression a crack closes (except perhaps for small regions near the crack tip) and stress concentration at the crack tip disappears.

Comparison with experiment. In spite of the large number of experimental studies of fatigue strength, the growth of fixed fatigue cracks has been investigated only in the last ten years. The first investigators in this field (Orowan, Head, Frost, Weibull) failed to appreciate the local character of the laws of rupture at the crack tip and, as a result, formulated their results in noninvariant variables. This local character became apparent after the work of Irwin. Extensive experimental investigations of the rate of growth of fatigue cracks have recently been carried out by Donaldson and Anderson [28], Paris [26], and Pearson [29] who studied numerous aluminum, molybdenum, titanium and other metal alloys at $N_{min} = 0$. Paris obtained the formula $dl/dn \sim N_{max}^4$, while the formula $dl/dn \sim N_{max}^{3.6}$ was obtained by Pearson. In a previous work Liu, obviously influenced by erroneous theories of Frost and Dugdale [6], derived an expression $dl/dn \sim N_{max}^2$; Paris pointed out inaccuracies in his calculations which were admitted by Liu in the ensuing discussion [26].

In accordance with (2.7), at $N_{max}/K_c \leqslant 0.5$ it is possible to replace (accurate to about 15%) Eq. (2.6) by the Paris formula. If one bears in mind the wide statistical scatter of experimental data, such an agreement between our results and the experimental results obtained by Paris and Pearson may be regarded as quite satisfactory.

Formula (2.5) describes also quite accurately the data of Donaldson and Anderson [28]. Markochev* found an exponential relationship $dl/dn \sim A + \exp(BN_{max})$ for four alloys studied in the range of a relatively small number of cycles (10^3–10^4), i. e., at N_{max}/K_c approaching 1.0 (low-endurance fatigue). These results are also satisfactorily described by Eq. (2.5). The existence of different empirical formulas is attributable to a wide statistical scatter of test results and to the fact that different investigations were carried out in different ranges of the N_* versus Δl_* diagram.

The results of this investigation can be easily applied to shear cracks. An approximate theoretical relation $dl/dn \sim N_{max}^4$ was derived for longitudinal shear cracks by McClintock, who based his derivation on the theory of accumulation of plastic strains in the plastic region of the material [28].

3. NONPROPAGATING FATIGUE CRACKS

Experiment shows [6–8] that fatigue cracks produced in the initial fatigue stages sometimes do not grow regardless of the number of loading cycles. In the physical sense this effect is obviously associated with the

*V. M. Markochev, Dissertation: "Methods of Investigating the Kinetics of Macrofracture of Sheet Materials Under Single and Repeated Loads" [in Russian], VIAM, Moscow, 1966.

microheterogeneity and grain structure of real materials and with their adaptability characteristics.* The latter conclusion can easily be reached starting from the general nonvariance considerations for cracks that satisfy the formulation of the problem of the fine structure of a crack tip when the concept of stress intensity coefficient is meaningful. In fact, in the most general case the condition of nonpropagation of a crack tip at $N = N_{max}$ and at a very large number of loading cycles (when the effect of the initial conditions may be neglected) can be written in the form of a certain inequality, in which parameters of the elastic-plastic medium (stresses, strains, displacements, etc.) and their functional characteristics near the crack tip (in view of the local character of rupture) appear. Since all the parameters of the medium near the crack tip at large n and at $N = N_{max}$ depend only on N_{max} and N_{min}, any given inequality will be reduced to

$$N_{max} \leqslant K_Y f(N_{min}/N_{max}) \qquad (0 \leqslant K_Y \leqslant K_c). \tag{3.1}$$

Here f is a certain dimensionless function of its parameter, and K_Y is a material constant. If $K_Y \neq 0$, the fatigue limit of the material will obviously be not equal to zero.

Extensive experimental studies aimed at the determination of the condition of nonpropagation of a crack emerging, in a special case, on the free rectilinear boundary of a "semi-infinite" plate in a direction normal to the boundary (the plate being subjected at "infinity" to a uniform cyclic tensile stress p with a constant coefficient of cycle asymmetry) were carried out by Frost [30, 31]. This work led to the formulation of an empirical condition $p_{max}^3 l < C$, where C is a material constant. On the basis of considerations of dimensional analysis N_{max}^2 in this case is equal to $\lambda p_{max}^2 l$, where λ is a certain number. Hence, on the basis of the general condition (3.1) a crack will not grow if $p_{max}^2 l < K_Y/\lambda$. This must be regarded as being in satisfactory agreement with Frost's results, especially if it is borne in mind that in Frost's experiment at small l the necessary condition $d \ll l$ (fine structure of the crack tip) was only partly satisfied.

It should be noted that Irwin's condition $N = K_c$ for brittle cracks can be obtained from the same nonvariance considerations.

4. STABILITY OF CRACK PROPAGATION

Investigations of conditions of stable crack propagation are of considerable importance because the transition to the unstable range leads in practice to a failure of a given structure.

Monotonic increase in the load. Let p denote the external load parameter and l the crack length parameter, which is a certain function of p. Let us assume that the external load and crack length increase with increasing p and l, respectively. The parametric condition of the stability of growth of the crack tip will then be in the form

$$\frac{dp}{dl} > 0. \tag{4.1}$$

Changing the sign in (4.1) will give the condition of instability. Let us find the form of (4.1) in the case of a monotonically increasing load acting on an elastic-plastic body. Since the function $N = N(p, l)$ is determined from the results of elastic analysis of stresses, the dependence of p on l is given in an implicit form by (1.8). Differentiating it in respect to l and using (4.1), we obtain the following condition of stability:

$$\left[\frac{\alpha^2 \sigma_s^3 (K_c^2 - N^2)}{\alpha_3 E N^3} - \frac{\partial N}{\partial l}\right] \Big/ \frac{\partial N}{\partial p} > 0. \tag{4.2}$$

The condition of instability is obtained by changing the sign in (4.2). Inequality (4.2) makes it possible (on the basis of elastic analysis of the problem) a priori to determine the regions of stable and unstable crack propagation on the surface (p, l).

In a special case, at $N = K_c$ we obtain from (4.2) the following condition of the stability of crack propagation:

$$\frac{\partial N}{\partial l} \Big/ \frac{\partial N}{\partial p} < 0. \tag{4.3}$$

*V. V. Bolotin drew the author's attention to the role played by adaptability in the problem under consideration.

Cyclic loads. In accordance with (2.6) the length of a crack under the influence of cyclic loads always monotonically increases with increasing number of stress cycles ($dl/dn > 0$). Considering the increase in applied load during a single cycle and using formulas (2.2) and (2.3) and condition (4.1), we obtain—in the same way as before—the following condition of stability:

$$\left[\frac{K_c^2 (K_c^2 - N^2)}{2\beta N^3} - \frac{\partial N}{\partial l}\right] \Big/ \frac{\partial N}{\partial p} > 0. \qquad (4.4)$$

The condition of instability is obtained by changing the sign in (4.4). In accordance with (4.4) the number of cycles before the onset of instability (which usually leads to the failure of a given structure) is given by the equation

$$K_c^2 (K_c^2 - N^2) = 2\beta N^3 \frac{\partial N}{\partial l} \qquad (N = N(p, l)). \qquad (4.5)$$

in which we substitute $p = p_{max}$ and function $l = l(p_{max}, p_{min}, n)$ found from a solution of the differential equation (2.5).

Finally, it should be noted that the solution obtained from (2.5) has a limit $N(p, l) \leqslant K_c$. When this limit is approached, the crack propagation rate approaches infinity (though the crack length is finite).

5. CONCRETE PROBLEMS

Let us consider some most typical problems of the propagation of fatigue cracks.

An analysis of the Griffith problem. Let an infinite plate pulled at infinity by uniaxial stresses $\sigma_y = p$ have a through rectilinear crack of a length $2l$ (Fig. 4). The applied stress is normal to the crack line. In this case the stress intensity coefficient is given by [14]

$$N = p\sqrt{l}/\sqrt{2}. \qquad (5.1)$$

Let us assume that the load p is a certain periodic function of time ($p_{max} \geqslant p \geqslant p_{min}$).

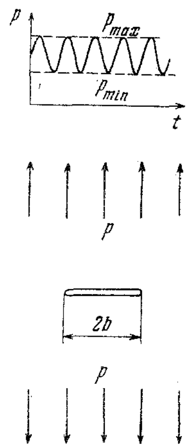

Fig. 4

In accordance with (2.5) and (5.1) the crack propagation rate is given by

$$\frac{dl}{dn} = -\beta\left[(a - b)\,l + \ln\frac{1 - al}{1 - bl}\right]$$
$$\left(a = \frac{p_{max}^2}{2K_c^2}, \quad b = \frac{p_{min}^2}{2K_c^2}\right). \qquad (5.2)$$

At $p_{min} < 0$ one should take $b = 0$.

The number of cycles at the instant of the loss of stability n_f is determined with the aid of (5.1), (5.2), and (4.5) from the following set of equations:

$$a l(\beta a+1)=1, \qquad \beta a n_f=\int_{a l_0}^{a l} \frac{dx}{\ln(1-bx/a)-\ln(1-x)-(1-b/a)x}. \tag{5.3}$$

Here l_0 denotes the initial crack length.

The solution of (5.2) exists in the range $0 < n < n_c$, $l_0 < l < a^{-1}$, where the maximum possible number of cycles n_c is given by the second equation in (5.3) at $al = 1$, in which it is only necessary to substitute n_f for n_c. It is obvious that the boundary of the existence of the solution $n = n_c$ defines the onset of brittle fracture corresponding to the Griffith solution $p^2_{max} = 2K^2_c l$ [32]. The latter corresponds to the instant of rapid acceleration of crack propagation.

In accordance with (4.4), (5.1), and (5.3) at $n_c > n > n_f$ the crack will propagate under unstable conditions; in these circumstances n_f should be regarded as the number of cycles to rupture of the plate as a whole.

In accordance with (5.2) the function $l = l(n)$ monotonically increases from $l = l_0$ to $l = a^{-1} = l(n_c)$; at the same time, $l'(n_c) = \infty$. Values of this function at $al_0 = 0.1$ and $b = 0$ are given below for certain values of dimensionless variables $l_* = al$, $n_* = a\beta n$.

$n_*=1$	4	6	8	10	12	14	15
$l_*=0.10$	0.12	0.15	0.18	0.22	0.30	0.55	0.82

Below we reproduce values of p_{max} calculated from (5.3) for different numbers of cycles n_f at $b = 0$ and $\beta p^2_{max} \ll 2K^2_c$ (in dimensionless variables $a_* = al_0$, $n_{f*} = \beta n_c/l_0$).

$n_{f*}=0$	2	4	6	10	20	160
$a_*=0.80$	0.48	0.40	0.35	0.29	0.25	0.20

The fatigue limit in this case is approximately $K^2_Y \approx 0.2K^2_c$.

On the basis of (2.4) and (5.1) the condition $\beta p^2_{max} \ll 2K^2_c$ means that the increase in the crack length per cycle is small in comparison with the total crack length.

It is known that standard tensile tests on elastic-plastic materials give highly reproducible results, while the results of fatigue tests on the same materials are as a rule widely scattered. This fact is attributable, in the theory postulated, to the presence of parameter l_0 in formula (5.3); this parameter represents the length of cracks initially present in the material or formed in the initial fatigue stages and is evidently a stochastic material constant.

Thus, the statistical nature of fatigue strength is analogous to the statistical nature of brittle strength.

A crack emerging on the body surface. Let a rectilinear crack of a length l emerge on a free flat boundary of a half-space in a direction normal to this boundary. It is assumed that the conditions of plane strain or plane stress state are satisfied. It is assumed also that the crack edges are not under load and that cyclic stresses p parallel to the boundary of the half-space are applied at infinity. In this case the stress intensity coefficient is [33]

$$N = 0.79\, p\sqrt{l} \tag{5.4}$$

All the qualitative singularities characteristic of the preceding problem are retained in this problem. In particular this applies to formulas (5.2) and (5.3) for the crack propagation rate and the ultimate number of cycles; calculations carried out above also remain valid, a and b being given by the following expressions:

$$a = 0.62\frac{p^2_{max}}{K_c^2}, \qquad b = 0.62\frac{p^2_{min}}{K_c^2} \tag{5.5}$$

A crack under the influence of a concentrated force. Let the opposite edges of a through rectilinear crack of a length $2l$ in an infinite plate be acted on by two equal and opposite concentrated forces P. The forces are applied in the center of the crack normal to its surface; there are no stresses applied at infinity. In this case the stress intensity coefficient is

$$N = \frac{P}{\pi\sqrt{2l}}. \tag{5.6}$$

Let P be a periodic function of time ($p_{max} \geqslant P \geqslant 0$). In accordance with (2.5) and (5.6) the crack propagation rate will be

$$\frac{dl}{dn} = -\beta \left[\frac{P^2_{max}}{2\pi^2 K_c^2 l} + \ln \left(1 - \frac{P^2_{max}}{2\pi^2 K_c^2 l} \right) \right]. \tag{5.7}$$

In this case a fatigue crack begins to grow when the initial crack length l_0 is increased to l_1 under the influence of the load monotonically increasing to P_{max} during the first loading cycle. In accordance with (5.6) and (1.8) the length l_1 is found from

$$l_1 = l_0 - \frac{\alpha_3 E K_c^2}{2\alpha_2 \sigma_s^3} \left[\frac{P^2_{max}}{2\pi^2 K_c^2 l_1} + \ln \left(1 - \frac{P^2_{max}}{2\pi^2 K_c^2 l_1} \right) \right] \tag{5.8}$$

which is easily solved with the aid of the graph in Fig. 3.

The solution of Eq. (5.7) can be represented in the form

$$n_* = - \int_{1/l_*}^{1/l_{*1}} \frac{dx}{x^2 [x + \ln (1 - x)]} \tag{5.9}$$

$$\left(n_* = \frac{2\pi^2 K_c^2 \beta}{P^2_{max}} n, \quad l_* = \frac{2\pi^2 K_c^2}{P^2_{max}} l, \quad l_{*1} = \frac{2\pi^2 K_c^2}{P^2_{max}} l_1 \right)$$

Certain values of function $l_* = l_*(n_*)$ at $l_{*1} = 1$ are given below:

n_*=0	1	2	4	8	12
l_*=1.00	1.60	1.75	2.31	2.81	3.00

In accordance with the general condition for nonpropagation of cracks (3.2), the crack under consideration will grow to a length l_Y

$$l_Y = \frac{P^2_{max}}{2\pi^2 K^2_Y}, \tag{5.10}$$

after which its growth will cease (at $l_{*1} = 1$, $l_{Y*} \approx 3$ and $K_Y^2 \approx K_c^2/3$).

A **crack in a rectangular cross section beam (Fig. 5).** Let a beam of rectangular cross section be subjected to pure alternating bending by a moment M ($M_{max} \geqslant M \geqslant -M_{max}$). In this case the stress intensity coefficient in the tips of symmetrically growing cracks will be [34]

$$N = \frac{M}{L^{3/2}} \sqrt{f(\lambda)} \qquad \left(\lambda = \frac{l}{L} \right). \tag{5.11}$$

Some values of function f are given below:

λ =0	1	2	4	6	10	15	20
$f(\lambda)$=0	0.11	0.20	0.32	0.39	0.47	0.53	0.57

In accordance with (2.5) the crack propagation rate is

$$\frac{d\lambda}{dn_*} = -M^2_* f(\lambda) - \ln [1 - M^2_* f(\lambda)] \tag{5.12}$$

$$\left(M_* = \frac{M}{K_c L^{3/2}}, \quad n_* = \frac{\beta}{L} n \right).$$

The solution of (5.12) is easily found with the aid of graphical integration. The function λ depends on n_*, on the initial value λ_0, and on M_*. The ultimate number of cycles to rupture is, in accordance with (4.5) and (5.11), given by

$$M_* \sqrt{f[\lambda(n_*, \lambda_0, M_*)]} = 1, \tag{5.13}$$

if the increase in the crack length per cycle is negligibly small in comparison with the beam width. As a numerical example the following figures can be cited: at $\lambda_0 = 0.05$ and $M_* = 0.33$ we have $\lambda_{max} \approx 0.4$ and $n_{f*} = 10$.

Several values of the function $\lambda(n_*)$ are given below:

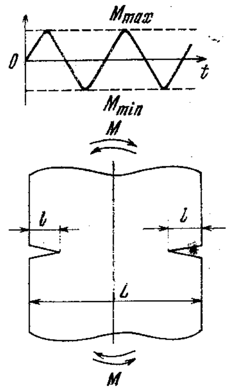

Fig. 5

$n_*=0$	2	5	7	8	9	9.5
$\lambda=0.05$	0.06	0.08	0.11	0.15	0.25	0.36

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